A Stepwise Planned Approach to the Solution of Hilbert's Sixth Problem. III : Measurements and von Neumann Projection/Collapse Rule

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Supmech, the universal mechanics developed in the previous two papers, accommodates both quantum and classical mechanics as subdisciplines (a brief outline is included for completeness); this feature facilitates, in a supmech based treatment of quantum measurements, an unambiguous treatment of the apparatus as a quantum system approximated well by a classical one. Taking explicitly into consideration the fact that observations on the apparatus are made when it has 'settled down after the measurement interaction' and are restricted to macroscopically distinguishable pointer readings, the unwanted superpositions of (system + apparatus) states are shown to be suppressed; this provides a genuinely physics based justification for the (traditionally *postulated*) von Neumann projection/collapse rule. The decoherence mechanism brought into play by the stated observational constraints is free from the objections against the traditional decoherence program.

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1. Introduction

In this open-ended program aimed at a solution of Hilbert's sixth problem (relating to axiomatization of physics and probability theory), the first two papers (Dass [1,2]; henceforth referred to as I and II) were devoted to evolving the geometro-statistical framework of a universal mechanics called 'supmech' and a consistent autonomous treatment of quantum systems in that framework. In this third paper, we shall treat measurements on quantum systems in the supmech framework and obtain a straightforward derivation of the von Neumann projection/collapse rule, obtaining, in the process, a clear understanding of the sense in which this rule should be understood.

The negative result about the possibility of a consistent quantum-classical interaction in the supmech framework [obtained as part of theorem (2) in I; see section 4.4 below] is by no means 'fatal' for a consistent treatment of measurement interaction between the system and apparatus. It turns out that it is adequate to treat the apparatus carefully as a quantum system approximated well by a classical one (in the setting of, for example, phase space descriptions of quantum and classical dynamics); the fact that both quantum and classical mechanics are special subdisciplines of supmech is very helpful in such a treatment. We shall see that, taking properly into consideration (i) the 'settling down of the apparatus after the measurement interaction', and (ii) the fact that the observations on the apparatus are restricted to macroscopically distinguishable pointer readings (this is what *automatically* brings into play the decohering effect of the internal environment of the apparatus), the unwanted superpositions of (system + apparatus)-states can be shown to be suppressed, leading eventually to the projection/collapse rule postulated in von Neumann's treatment of measurements [3].

In the next section, the measurement problem in quantum mechanics (QM) is recalled. In section 3, some proposed improvements in the treatment of the physics of the apparatus are outlined. In section 4, we briefly recall the essential features of supmech and, in section 5, a supmech-based treatment of measurements on a quantum system is given leading eventually to the von Neumann projection rule; the functioning of some crucial ingredients of this treatment is illustrated with the example of the Stern-Gerlach experiment in section 6. In section 7, the present work is compared with the traditional decoherence program. In section 8, we add, to the list of seven axioms of the supmech program given in II, another one covering measurement situations. The last section contains some concluding remarks.

2. The Measurement Problem in Quantum Mechanics

We consider, for simplicity, the measurement of an observable (of a quantum system S) represented by a self-adjoint operator F (acting in an appropriate dense domain in the Hilbert space $\mathcal{H}_S$ of S) having a non-degenerate spectrum with the eigenvalue equations $F|\psi_j>=\lambda_j|\psi_j>$ (j = 1,2,...). The apparatus A is chosen such that, to each of the eigenvalues λ_j corresponds a pointer position M_j. If the system is initially in an eigenstate $|\psi_j>$, the apparatus is supposedly designed to give, after the measurement interaction, the pointer reading M_j; the outcome of the measurement is then understood as λ_j. A question immediately arises : 'What is the measurement outcome when the initial state of the system S is a superposition state $|\psi>=\sum_j c_j|\psi_j>$?' The theoretical framework employed for the treatment of measurements on quantum systems must provide a satisfactory answer to this question.

The standard treatment of measurements in QM (von Neumann [3]; Wheeler and Zurek [4]; Jauch [5]; Omnes [6]; Dass [7]) is due to von Neumann who, treating the apparatus as a quantum system, introduced, for the pointer positions M_j, state vectors $|\mu_j>$ in the Hilbert space $\mathcal{H}_A$ of the apparatus. The Hilbert space for the coupled system (S + A) is taken to be the tensor product $\mathcal{H}=\mathcal{H}_S\otimes\mathcal{H}_A$. The measurement interaction is elegantly described (Omnes [6]; Dass [7]) by a unitary operator U on $\mathcal{H}$ which, acting on the initial state of (S+A) (with the system S in the initial state in which it is prepared for the experiment and the apparatus in the 'ready' state which we denote as $|\mu_0>$) gives an appropriate final state. We shall assume the measurement to be *ideal* which is supposedly such that (Omnes [8]) 'when the measured system is initially in an eigenstate of the measured observable, the measurement leaves it in the same state.' In this case, the measurement outcome must be the corresponding eigenvalue which must be indicated by the final pointer position. This implies

$$U(|\psi_j>\otimes|\mu_0>)=|\psi_j>\otimes|\mu_j>\,. \tag{1}$$

For S in the initial state $|\psi>=\sum c_j|\psi_j>$, the final (S + A)- state must be, by linearity of U,

$$|\Psi_f>\equiv U[(\sum_j c_j|\psi_j>)\otimes|\mu_0>]=\sum_j c_j[|\psi_j>\otimes|\mu_j>]. \tag{2}$$

Note that the right hand side of Eq.(2) is a superposition of the quantum states of the (generally *macroscopic*) system (S + A).

Experimentally, however, one does not observe such superpositions. Instead, one obtains, in each measurement, a definite outcome λ_j corresponding to the final (S + A)-state $|\psi_j > \otimes |\mu_j >$. Repetitions of the experiment, with system in the same initial state, yield various outcomes randomly such that, when the number of repetitions becomes large, the relative frequencies of various outcomes tend to have fixed values. To account for this, von Neumann postulated that, after the operation of the measurement interaction as above, a discontinuous, noncausal and instantaneous process takes place which changes the state $|\Psi_f >$ to the state represented by the density operator

$$\rho_f = \sum_i \tilde{P}_i |\Psi_f >< \Psi_f| \tilde{P}_i \tag{3}$$

$$= \sum_j |c_j|^2 [|\psi_j >< \psi_j| \otimes |\mu_j >< \mu_j|]; \tag{4}$$

here $\tilde{P}_i = |\psi_i >< \psi_i| \otimes I_A$ where I_A is the identity operator on $\mathcal{H}_A$. This is referred to as von Neumann's *projection postulate* and the phenomenon with the above process as the underlying process the *state vector reduction* or *wave function collapse*. Eq.(4) represents, in the von Neumann scheme, the (S +A)-state on the completion of the measurement. It represents an ensemble of (S + A)-systems in which a fraction $p_j = |c_j|^2$ appears in the j th product state in the summand. With the projection postulate incorporated, the von Neumann formalism, therefore, predicts that, in a measurement with the system S initially in the superposition state as above,
(i) the measured values of the observable F are the random numbers λ_j with respective probabilities $|c_j|^2$;
(ii) when the measurement outcome is λ_j, the final state of the system is $|\psi_j >$.
Both the predictions are in excellent agreement with experiment.

The main problem with the treatment of a quantum measurement given above is the ad-hoc nature of the projection postulate. Moreover, having to invoke a discontinuous, acausal and instantaneous process is an unpleasant feature of the formalism. The so-called measurement problem in QM is essentially the problem of explaining the final state (4) without introducing anything ad-hoc and/or physically unappealing in the theoretical treatment. This means that one should either give a convincing dynamical explanation of the reduction process or else circumvent it; we shall do the former.

A critical account of various attempts to solve the measurement problem and related detailed references may be found in the author's article [7]; none of them can be claimed to have provided a satisfactory solution. [Even the relatively more appealing decoherence program (Zurek [9]) has problems ([10,11,7]; see section 7).]

3. Doing Justice to the Physics of the Apparatus

Von Neumann's treatment does not do adequate justice to the physics of the apparatus and needs some improvements. We propose to take into consideration the following points :

(i) The apparatus A is a quantum mechanical system admitting, to a very good approximation, a classical description. Even when the number of the effective apparatus degrees of freedom is not large (for example, in the Stern-Gerlach experiment, treated in section 6, where the center of mass position vector of a silver atom acts as the effective apparatus variable), a classical description of the relevant variables is adequate. This feature must be properly incorporated in the theoretical treatment to obtain a satisfactory description of measurements. [Items (iii)-(v) below cannot be treated properly unless this feature is incorporated.] This is the right approach to avoid problems relating to 'quantum-classical divide' in the treatment of measurements on quantum systems.

(ii) Introduction of vector states for the pointer positions is neither desirable (no operational meaning can be assigned to a superposition of the pointer states $|\mu_j>$) nor necessary : a better procedure is to introduce density operators for the pointer states and take into consideration the fact that the Wigner functions corresponding to them are approximated well by classical phase space density functions.

(iii) The pointer states have a stability property : After the measurement interaction is over, the apparatus, left to itself, settles quickly into one of the pointer positions. It is this process which should replace von Neumann's 'instantaneous, non-causal and discontinuous' process.

Note. A detailed mathematical treatment of this process, as we shall see below, is not necessary; it is adequate to take its effect correctly into account. (The von Neumann projection postulate does this, but that is not enough; one must give a physics based justification for the prescription.) To get a feel

for this, note that, in, for example, the Stern-Gerlach experiment, treated in section VI, the measurement interaction is over (ignoring fringe effects) after the atom is out of the region between the magnetic pole pieces. In this case, by 'the apparatus settling to a pointer position' one means the movement of the atom from just outside the pole pieces to a detector. In this case, the choice of the detector is decided by the location of the atom just after the measurement interaction. Details of motion of the atom from the magnets to the detector is of no practical interest in the present context. In the case of a macroscopic apparatus, the 'settling ...' refers to the process of the apparatus reattaining (thermal, mechanical) equilibrium (disturbed slightly during the measurement interaction) after the measurement interaction; again, details of this process are not important in the present context. Different pointer positions supposedly have [see axiom A8(d) in section 8] mutually disjoint stability domains in the phase space of the apparatus. The eventual pointer position indicated in an experiment is decided by the stability domain in which the point representing the classical state of the apparatus happens to be immediately after the measurement interaction.

(iv) Observations relating to the apparatus are restricted to the pointer positions M_j. A properly formulated dynamics (classical or quantum) which takes this into consideration (treating the apparatus 'respectfully' as a *system*) would involve, at appropriate stage, averaging over the inoperative part of the phase space of the apparatus. It is this averaging, as we shall see below, which [combined with item (v) below] produces the needful decoherence effects to wipe out undesirable quantum interferences.

(v) Different pointer positions are macroscopically distinguishable. We shall take this into consideration by employing an appropriate energy-time uncertainty inequality.

4. Supmech : A Brief Outline

Supmech is an 'all-embracing' mechanics having both classical and quantum mechanics as its subdisciplines. Its framework facilitates an autonomous development of QM (i.e. without having to quantize classical dynamical systems) and a transparent treatment of quantum-classical correspondence. A brief presentations of its basic features follows. Since fermionic objects are not needed in the present work, we shall present only the bosonic version of supmech.

4.1 Probabilistic framework

a. Experimentally accessible systems. By these, we mean systems whose 'identical' (for all practical purposes) copies are reasonably freely available for repeated trials of an experiment. Henceforth by a system we shall mean an experimentally accessible one. Some aspects of systems not included in this class (the universe and its 'large' subsystems) can be covered by the formalism of this paper with the slightly more refined presentation of the basic postulates as given in II and an appropriate interpretation of classical probabilities in the statistical analysis of the experimental data relating to such systems.

b. System algebra; Observables. Supmech associates, with every system S, a complex associative topological $\star$-algebra $\mathcal{A} = \mathcal{A}^{(S)}$ having a unit element (denoted here as I). [The topology is assumed to be locally convex with some additional features (described in I); we shall skip the details.] Observables of S are elements of the subset $\mathcal{O}(\mathcal{A})$ of Hermitian elements of $\mathcal{A}$. A positive observable is a sum of terms of the form $\sum_i A_i^* A_i$ where $A_i \in \mathcal{A}$.

c. States. States of the system, also referred to as the states of the system algebra $\mathcal{A}$ (denoted by the letters $\phi, \phi', ..$), are defined as continuous positive linear functionals on $\mathcal{A}$ which are normalized [i.e. $\phi(I) = 1$]. The set of states of $\mathcal{A}$ will be denoted as $\mathcal{S}(\mathcal{A})$ and the subset of pure states (i.e. those not expressible as nontrivial convex combinations of other states) by $\mathcal{S}_1(\mathcal{A})$. For any $A \in \mathcal{O}(\mathcal{A})$ and $\phi \in \mathcal{S}(\mathcal{A})$, the quantity $\phi(A)$ is to be interpreted as the expectation value of A when the system is in the state ϕ.

d. Compatible completeness of observables and pure states. The pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$ is assumed to be *compatibly complete* in the sense that

(i) given $A, B \in \mathcal{O}(\mathcal{A}), A \neq B$, there should be a state $\phi \in \mathcal{S}_1(\mathcal{A})$ such that $\phi(A) \neq \phi(B)$;

(ii) given two different states ϕ_1 and ϕ_2 in $\mathcal{S}_1(\mathcal{A})$, there should be an $A \in \mathcal{O}(\mathcal{A})$ such that $\phi_1(A) \neq \phi_2(A)$.

We shall refer to this condition as the 'CC condition' for the pair $(\mathcal{O}(\mathcal{A}), \mathcal{S}_1(\mathcal{A}))$.

e. Experimental situations and probabilities. In supmech, experimental situations relating to a system with system algebra $\mathcal{A}$ are formalized in terms of *positive observable valued measures* (PObVMs) defined as follows. One introduces a measurable space $(\Omega, \mathcal{F})$ where Ω is the 'value space' (spectral

space) of one or more observables and elements of $\mathcal{F}$ (measurable subsets) are (standardized idealizations of) those subsets of Ω which can be experimentally distinguished. A POVM for the system S, based on this measurable space, is a family $\{\nu(E); E \in \mathcal{F}\}$ where the objects $\nu(E)$ (*supmech events*) are positive observables in $\mathcal{A}$ such that

$$(i)\ \nu(\emptyset) = 0,\ (ii)\ \nu(\Omega) = I,$$
$$(iii)\ \nu(\cup_i E_i) = \textstyle\sum_i \nu(E_i)\ \text{(for disjoint unions)}.$$

It is the abstract counterpart of the 'positive operator-valued measure' (POVM) employed in Hilbert space QM. Given a state ϕ of the system S, we have a probability measure p_ϕ on $(\Omega, \mathcal{F})$ given by

$$p_\phi(E) = \phi(\nu(E)) \quad \forall E \in \mathcal{F}. \tag{5}$$

Eq.(5) represents the theoretically desirable relationship between supmech expectation values and classical probabilities. In supmech, all probabilities in the formalism (which relate to statistics of measurement outcomes) are stipulated to be of this type (i.e. expectation values of supmech events).

4.2 Noncommutative symplectic geometry

a. Derivation based noncommutative differential calculus (Dubois-Violette [12,13]). Replacing, in the classical differential form calculus, the commutative algebra $C^\infty(M)$ of smooth complex valued functions on a manifold M by an algebra $\mathcal{A}$ in the class mentioned above and the Lie algebra $\mathcal{X}(M)$ of smooth complex vector fields on M by the Lie algebra $Der(\mathcal{A})$ of derivations of $\mathcal{A}$, the elements of the space $\Omega^p(\mathcal{A})$ of noncommutative differential p-forms on $\mathcal{A}$ (p= 1,2,..) are defined as multilinear maps $(Der(\mathcal{A}))^p \to \mathcal{A}$ such that, for $\omega \in \Omega^p(\mathcal{A}),\ X, Y \in Der(\mathcal{A})$ and $K \in Z(\mathcal{A})$, the center of $\mathcal{A}$, we have

$$\omega(.., X, Y, ..) = -\ \omega(.., Y, X, ..);\quad \omega(.., KX, ..) = K\omega(.., X, ..);$$

moreover, $\Omega^0(\mathcal{A}) = \mathcal{A}$. The usual operations on differential forms [exterior product $\wedge : \Omega^p(\mathcal{A}) \times \Omega^q(\mathcal{A}) \to \Omega^{p+q}(\mathcal{A})$, exterior derivative $d : \Omega^p(\mathcal{A}) \to \Omega^{p+1}(\mathcal{A})$, interior product $i_X : \Omega^p(\mathcal{A}) \to \Omega^{p-1}(\mathcal{A})$ and Lie derivative $L_X : \Omega^p(\mathcal{A}) \to \Omega^p(\mathcal{A})$ are defined along lines parallel to the commutative case and analogous relations involving these operations hold, with very few exceptions. [The not so well known *algebraic* definition of the exterior derivative in the commutative case (Matsushima [14]) works as such for the noncommutative

case.] A differential form α is said to be closed if $d\alpha = 0$ and exact if $\alpha = d\beta$ for some form β.

b. Induced mappings on derivations and differential forms. These are analogues of the push-forward and pull-back mappings on vector fields and differential forms induced by diffeomorphisms between manifolds. Given a topological $\star$-algebra isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, we have the induced linear mappings $\Phi_* : Der(\mathcal{A}) \to Der(\mathcal{B})$ and $\Phi^* : \Omega^p(\mathcal{B}) \to \Omega^p(\mathcal{A})$ given by

$$\begin{aligned} (\Phi_* X)(B) &= \Phi(X[\Phi^{-1}(B)]) \text{ for all } X \in Der(\mathcal{A}) \text{ and } B \in \mathcal{B}; \\ (\Phi^*\omega)(X_1, .., X_p) &= \Phi^{-1}[\omega(\Phi_* X_1, .., \Phi_* X_p)] \\ &\quad \text{for all } \omega \in \Omega^p(\mathcal{B}) \text{ and } X_1, .., X_p \in Der(\mathcal{A}). \end{aligned}$$

These mappings satisfy the relations [with $\Psi : \mathcal{B} \to \mathcal{C}$ and other obvious notation]

$$\begin{aligned} (\Psi \circ \Phi)_* &= \Psi_* \circ \Phi_*; \quad \Phi_*[X, Y] = [\Phi_* X, \Phi_* Y]; \ (\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*; \\ \Phi^*(\alpha \wedge \beta) &= (\Phi^*\alpha) \wedge (\Phi^*\beta); \ \Phi^*(d\alpha) = d(\Phi^*\alpha). \end{aligned}$$

c. Symplectic structures; Poisson brackets (Dubois-Violette [12,13]); *Canonical transformations.* The system algebra $\mathcal{A}$ is assumed to be equipped with a symplectic form ω which, by definition, is a closed 2-form which is non-degenerate in the sense that, for any $A \in \mathcal{A}$, there is a unique derivation $Y_A \in Der(\mathcal{A})$ such that

$$i_{Y_A}\omega = -dA.$$

The pair $(\mathcal{A}, \omega)$ is called a *symplectic algebra*. For any two elements A,B of $\mathcal{A}$, their Poisson bracket (PB) is defined as $\{A, B\} = Y_A(B)$ and has the usual properties of bilinearity, antisymmetry, Leibnitz rule and Jacobi identity.

Given two symplectic algebras $(\mathcal{A}, \omega)$ and $(\mathcal{A}', \omega')$, a topological $\star$-algebra isomorphism $\Phi : \mathcal{A} \to \mathcal{A}'$ is called a *symplectic mapping* if $\Phi^*\omega' = \omega$. A symplectic mapping of $(\mathcal{A}, \omega)$ onto itself is called a canonical/symplectic transformation. An infinitesimal transformation of $\mathcal{A}$ of the form

$$A \mapsto A + \delta A; \ \delta A = \epsilon\{G, A\} \tag{6}$$

is a canonical transformation (generated by $G \in \mathcal{A}$).

d. Special algebras; the canonical symplectic form. An algebra $\mathcal{A}$ of the above mentioned type is called special if all its derivations are inner (i.e. those of

the form D_A with $D_A(B) = [A, B]$). The differential 2-form ω_c defined on such an algebra $\mathcal{A}$ by

$$\omega_c(D_A, D_B) = [A, B] \tag{7}$$

is said to be the *canonical form* on $\mathcal{A}$. It is a symplectic form giving, for $A, B \in \mathcal{A}$, $Y_A = D_A$ and $\{A, B\} = [A, B]$. If one takes, on such an algebra, the form $\omega = b\ \omega_c$ as a symplectic form (where b is a nonzero complex number), we have

$$Y_A = b^{-1} D_A, \qquad \{A, B\} = b^{-1}[A, B]. \tag{8}$$

The *quantum Poisson bracket*

$$\{A, B\}_Q = (-i\hbar)^{-1}[A, B] \tag{9}$$

is a special case of this with $b = -i\hbar$; the corresponding symplectic form is the *quantum symplectic form* $\omega_Q = -i\hbar\omega_c$.

4.3 Dynamics.

Dynamics in supmech is described (in the Heisenberg type picture) by a one-parameter family Φ_t of canonical transformations generated by by an observable $H \in \mathcal{O}(\mathcal{A})$ called the Hamiltonian. Writing $\Phi_t(A) = A(t)$ and taking G = H and $\epsilon = \delta t$ in Eq.(6), we have the *supmech Hamilton's equation*

$$\frac{dA(t)}{dt} = \{H, A(t)\} \equiv \partial_H(A(t)). \tag{10}$$

In the Schrödinger type picture, time evolution is carried by states, the two descriptions being related as

$$< \phi, A(t) >=< \phi(t), A > \tag{11}$$

[so that $\phi(t) = \tilde{\Phi}_t(\phi)$ where the tilde indicates transpose]. Writing $\phi(t+\delta t) = \phi(t) + \delta\phi(t)$, we have, from equations (10,11), the *supmech Liouville equation* for the time evolution of states :

$$\frac{d\phi(t)}{dt}(A) = \phi(t)(\{H, A\}) \equiv (\tilde{\partial}_H(\phi(t))(A). \tag{12}$$

We may write, formally,

$$\Phi_t = exp(t\partial_H); \quad \tilde{\Phi}_t = exp(t\tilde{\partial}_H). \tag{13}$$

The quadruple $\Sigma = (\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$ is called a *supmech Hamiltonian system*. Another supmech hamiltonian system Σ' is said to be equivalent to Σ if there is a symplectic mapping $\Phi : (\mathcal{A}, \omega) \to (\mathcal{A}', \omega')$ such that $\Phi(H) = H'$. The states are then related through $\tilde{\Phi}$. When states are not being considered, we may refer to a triple $(\mathcal{A}, \omega, H)$ as a supmech Hamiltonian system.

4.4 Interaction between two systems in supmech

Given two sytems S_1 and S_2 considered as supmech Hamiltonian systems $\Sigma_i = (\mathcal{A}^{(i)}, \mathcal{S}_1(\mathcal{A}^{(i)}), \omega^{(i)}, H^{(i)})$ [the PBs in the two algebras will be denoted as $\{.,.\}_i$ (i=1,2)], we treat the coupled system $(S_1 + S_2)$ as a supmech Hamiltonian system $\Sigma = (\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$ where $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$,

$$\omega = \omega^{(1)} \otimes I_2 + I_1 \otimes \omega^{(2)}, \tag{14}$$

$$H = H^{(1)} \otimes I_2 + I_1 \otimes H^{(2)} + H_{int} \tag{15}$$

where I_i is the unit element of $\mathcal{A}^{(i)}$ (i= 1,2) and, typically,

$$H_{int} = \sum_{i=1}^{n} F_i \otimes G_i.$$

The 2-form ω of (14) is closed but, according to theorem (2) in I, is non-degenerate if and only if either both the algebras $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are commutative or both noncommutative with their respective PBs proportional to commutators with the *same* proportionality constant i.e.

$$\{A, B\}_1 = i\lambda[A, B]; \quad \{C, D\}_2 = i\lambda[C, D] \tag{16}$$

(with the parameter λ nonzero and real; it can be chosen, by replacing the initially chosen symplectic form by its negative, if necessary, to be positive). We make the identification $\lambda = \hbar^{-1}$; the formalism, therefore, *dictates* the presence of a universal Planck type constant.

In both the cases, the PB in $\mathcal{A}$ can be expressed in the form [I, Eq.(99)]

$$\{A \otimes B, C \otimes D\} = \{A, C\}_1 \otimes \frac{BD + DB}{2} + \frac{AC + CA}{2} \otimes \{B, D\}_2. \tag{17}$$

The dynamics of the coupled system is governed, in the Heisenberg type picture, by the supmech Hamilton's equation [I, Eq.(101)]

$$\begin{aligned} \frac{d}{dt}[A(t) \otimes B(t)] &= \{H, A(t) \otimes B(t)\} \\ &= \{H^{(1)}, A(t)\}_1 \otimes B(t) + A(t) \otimes \{H^{(2)}, B(t)\}_2 \\ &\quad + \{H_{int}, A(t) \otimes B(t)\}. \end{aligned} \tag{18}$$

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4.5 Classical Hamiltonian mechanics and traditional Hilbert space quantum mechanics as subdisciplines of supmech

A classical Hamiltonian system (P, ω_{cl}, H_{cl}) [where P is the phase space which is a symplectic manifold with the classical symplectic form $\omega_{cl} \equiv \sum dp_\alpha \wedge dq^\alpha$ (in canonical coordinates) and H_{cl} is the classical Hamiltonian, a smooth real-valued function on P] is a special case of a supmech Hamiltonian system $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega, H)$ with $\mathcal{A} = \mathcal{A}_{cl} \equiv C^\infty(P; \mathbb{C})$, $\mathcal{S}_1(\mathcal{A}) = P$ (Dirac measures on the phase space P identified with points of P), $\omega = \omega_{cl}$ and H $= H_{cl}$; the supmech PBs are now the traditional classical PBs. The supmech Hamilton's equation (10) is now the classical Hamilton's equation. Representing states by probability densities in phase space, Eq.(12) goes over, in appropriate cases (for $P = \mathbb{R}^{2n}$, for example, after the obvious partial integrations), to the classical Liouville equation for the density function. The CC condition can be easily verified in this case (II, section 2.2). The supmech events are now the characteristic/indicator functions corresponding to the Borel subsets of P (which correspond to events in classical probability theory) (II, section 2.1).

To see the traditional Hilbert space QM as a subdiscipline of supmech, it is useful to introduce the concept of a *quantum triple* $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ where $\mathcal{H}$ is a complex separable Hilbert space, $\mathcal{D}$ a dense linear subset of $\mathcal{H}$ and $\mathcal{A}$ an Op*-algebra of operators based on $(\mathcal{H}, \mathcal{D})$. [Such an algebra is a family of operators which, along with their adjoints, map $\mathcal{D}$ into itself. The *-operation on the algebra is defined as the restriction of the Hilbert space adjoint on $\mathcal{D}$. These are the algebras of operators (not necessarily bounded) appearing in the traditional Hilbert space QM; for example, the operator algebra generated by the position and momentum operators in the Schrodinger representation for a nonrelativistic spinless particle (the Heisenberg algebra) belongs to this class, with $\mathcal{H} = L^2(\mathbb{R}^3)$ and $\mathcal{D} = \mathcal{S}(\mathbb{R}^3)$.]

Here we shall consider only the *standard quantum triples* by which we mean those in which (i) the algebra $\mathcal{A}$ is special in the sense described above, and (ii) $\mathcal{A}$ acts irreducibly on $(\mathcal{H}, \mathcal{D})$ [i.e. there does not exist a smaller quantum triple $(\mathcal{H}', \mathcal{D}', \mathcal{A})$ with $\mathcal{D}' \subset \mathcal{D}$, $\mathcal{A}\mathcal{D}' \subset \mathcal{D}'$ and $\mathcal{H}'$ is a proper subspace of $\mathcal{H}$]. The quantum triple associated with the Schrödinger representation for a non-relativistic spinless particle mentioned above satisfies these conditions.

With $\mathcal{A}$ special, one can define the quantum symplectic form $\omega_Q = -i\hbar\omega_c$

which gives the Poisson brackets of Eq.(9). With the $\mathcal{A}$-action irreducible, the space $\mathcal{S}_1(\mathcal{A})$ of pure states of $\mathcal{A}$ consists of vector states corresponding to normalized vectors in $\mathcal{D}$. Choosing an appropriate self adjoint element H of $\mathcal{A}$ as the Hamiltonian operator, we have a quantum Hamiltonian system $(\mathcal{A}, \mathcal{S}_1(\mathcal{A}), \omega_Q, H)$ as a special case of a supmech Hamiltonian system. The POVMs are now the traditional POVMs (positive operator-valued measures). It was shown in II that the Born probabilities in traditional quantum mechanics can always be expressed in the form (5).

With the quantum PBs of Eq.(9), the supmech Hamilton's equation (10) goes over to the traditional Heisenberg equation of motion. General states are represented by density operators ρ satisfying the condition $|Tr(\overline{\rho A})| < \infty$ for all observables A in $\mathcal{A}$ (where the overbar indicates closure of the operator). The CC condition holds in this case as well (II, section 2). Noting that $Tr(\overline{\rho_1 A}) = Tr(\overline{\rho_2 A})$ for all $A \in \mathcal{A}$ implies $\rho_1 = \rho_2$, Eq.(12) goes over to the von Neumann equation

$$\frac{d\rho(t)}{dt} = (-i\hbar)^{-1}[\rho(t), H]. \tag{19}$$

4.6 Quantum-classical correspondence

This feature of supmech (of accommodating both classical and quantum mechanics) facilitates a transparent treatment of quantum-classical correspondence. The strategy adopted in II was to start with a quantum Hamiltonian system, transform it to an isomorphic supmech Hamiltonian system involving phase space functions and $\star$-products (Weyl-Wigner-Moyal formalism) and show that, in this latter Hamiltonian system, the subclass of phase space functions in the system algebra which go over to smooth functions in the $\hbar \to 0$ limit yield the corresponding classical Hamiltonian system. The working of this strategy was demonstrated for the case of a spinless nonrelativistic particle. It was, however, clear that the treatment permitted trivial generalization to systems with phase space $\mathbb{R}^{2n}$. We collect below the $\mathbb{R}^{2n}$-analogues of some equations from section 4 of II. [The integrals in equations (20-26) below are over $\mathbb{R}^n$.]

Given a quantum triple $(\mathcal{H}, \mathcal{D}, \mathcal{A})$ where $\mathcal{H} = L^2(\mathbb{R}^n), \mathcal{D} = \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{A}$ an Op^*-algebra based on $(\mathcal{H}, \mathcal{D})$, we have, for any $A \in \mathcal{A}$ and ϕ, ψ normalized elements in $\mathcal{D}$,

$$(\phi, A\psi) = \int\int \phi^*(y) K_A(y, y') \psi(y') dy dy' \tag{20}$$

where the kernel K_A is a (tempered) distribution. The Wigner function A_W corresponding to A is defined as the function on R^{2n} given by

$$A_W(x,p) = \int exp[-ip.y/\hbar]K_A(x+\frac{y}{2}, x-\frac{y}{2})dy. \tag{21}$$

Given a density operator ρ on $\mathcal{H}$ such that $|Tr(\overline{A\rho})| < \infty$ for all $A \in \mathcal{A}$ and defining ρ_W as above, we have

$$Tr(\overline{A\rho}) = \int\int A_W(x,p)\rho_W(x,p)dxdp. \tag{22}$$

The Wigner function ρ_W is real but generally not non-negative.

Introducing, in $\mathbb{R}^{2n}$, the notations $\xi = (x,p)$, $d\xi = dxdp$ and $\sigma(\xi,\xi^{'}) = p.x^{'} - x.p^{'}$ (the symplectic form in $\mathbb{R}^{2n}$), we have, for A,B $\in \mathcal{A}$

$$\begin{aligned}(AB)_W(\xi) &= (2\pi)^{-6}\int\int exp[-i\sigma(\xi-\eta,\tau)]A_W(\eta+\frac{\hbar\tau}{4}). \\ &\quad .B_W(\eta-\frac{\hbar\tau}{4})d\eta d\tau \\ &\equiv (A_W \star B_W)(\xi).\end{aligned} \tag{23}$$

The associativity condition $A(BC) = (AB)C$ implies the corresponding condition $A_W \star (B_W \star C_W) = (A_W \star B_W) \star C_W$ in the space $\mathcal{A}_W$ of the Wigner functions corresponding to the elements of $\mathcal{A}$ which is a complex associative non-commutative, unital *-algebra (with the *star-product* of Eq.(23) as product and complex conjugation as involution) isomorphic (as a star-algebra) to $\mathcal{A}$. Under this isomorphism, the quantum symplectic form $\omega_Q = -i\hbar\omega_c$ on $\mathcal{A}$ goes over to the 2-form $\omega_W = -i\hbar\omega_c^W$ where ω_c^W is the canonical form on $\mathcal{A}_W$; this makes the pair $(\mathcal{A}_W, \omega_W)$ a symplectic algebra isomorphic to $(\mathcal{A}, \omega_Q)$. The corresponding PB on $\mathcal{A}_W$ is given by the *Moyal bracket*

$$\{A_W, B_W\}_M \equiv (-i\hbar)^{-1}(A_W \star B_W - B_W \star A_W). \tag{24}$$

For functions f, g in $\mathcal{A}_W$ which are smooth and such that $f(\xi)$ and $g(\xi)$ have no $\hbar-$dependence, we have, from Eq.(23),

$$f \star g = fg - (i\hbar/2)\{f,g\}_{cl} + O(\hbar^2). \tag{25}$$

The functions $A_W(\xi)$ will have, in general, some $\hbar$ dependence and the $\hbar \to 0$ limit may be singular for some of them. We denote by $(\mathcal{A}_W)_{reg}$ the subclass

of functions in $\mathcal{A}_W$ whose $\hbar \to 0$ limits exist and are smooth (i.e. C^∞) functions; moreover, we demand that the Moyal bracket of every pair of functions in this subclass also have smooth limits. This class is easily seen to be a subalgebra of $\mathcal{A}_W$ closed under Moyal brackets. Now, given two functions A_W and B_W in this class, if $A_W \to A_{cl}$ and $B_W \to B_{cl}$ as $\hbar \to 0$, then $A_W \star B_W \to A_{cl}B_{cl}$; the subalgebra $(\mathcal{A}_W)_{reg}$, therefore, goes over, in the $\hbar \to 0$ limit , to a subalgebra $\mathcal{A}_{cl}$ of the commutative algebra $C^\infty(\mathbb{R}^{2n}, \mathbb{C})$ (with pointwise product as multiplication). The Moyal bracket of Eq.(24) goes over to the classical PB $\{A_{cl}, B_{cl}\}_{cl}$; the subalgebra $\mathcal{A}_{cl}$, therefore, is closed under the classical Poisson brackets. The classical PB $\{,\}_{cl}$ determines the classical symplectic form ω_{cl}. In the $\hbar \to 0$ limit, therefore, we have the classical symplectic algebra $(\mathcal{A}_{cl}, \omega_{cl})$. In situations where $H_W \in (\mathcal{A}_W)_{reg}$ admitting the $\hbar \to 0$ limit H_{cl}, we have, in this limit, the classical Hamiltonian system $(\mathcal{A}_{cl}, \omega_{cl}, H_{cl})$. [This is the case, for example, for the Hamiltonian operator $H = (2m)^{-1}\mathbf{P}^2 + V(\mathbf{X})$ with V a smooth function.]

When the $\hbar \to 0$ limits of A_W and ρ_W on the right hand side of Eq.(22) exist (call them A_{cl} and ρ_{cl}), we have, in this limit,

$$Tr(\overline{A\rho}) \to \int\int A_{cl}(x,p)\rho_{cl}(x,p)dxdp. \tag{26}$$

The quantity ρ_{cl} can be shown to be non-negative (and, therefore, a genuine density function on the phase space $\mathbb{R}^{2n}$).

We shall make, in our treatment of measurements below, the fairly safe assumption that this strategy works for the apparatus treated as a quantum system. [See the axiom A8(b,c) in section 8.] This will enable us to exploit the fact that the apparatus admits a classical description to a very good approximation.

5. Treatment of a Quantum Measurement in Supmech

We shall now treat the (S +A) system in the framework of section IV D above treating both, the system S and the apparatus A, as quantum Hamiltonian systems. Given the two quantum triples $(\mathcal{H}_S, \mathcal{D}_S, \mathcal{A}_S)$ and $(\mathcal{H}_A, \mathcal{D}_A, \mathcal{A}_A)$ corresponding to S and A, the quantum triple corresponding to (S+A) is $(\mathcal{H}_S \otimes \mathcal{H}_A, \mathcal{D}_S \otimes \mathcal{D}_A, \mathcal{A}_S \otimes \mathcal{A}_A)$.

A general pointer observable for A is of the form

$$J = \sum_j b_j P_j \tag{27}$$

where P_j is the projection operator onto the space of states in $\mathcal{H}_A$ corresponding to the pointer position M_j [considered as an apparatus property; for a detailed treatment of the relationship between classical properties and quantum mechanical projectors, see (Omnes [6,8]) and references therein] and b_js are real numbers such that $b_j \neq b_k$ for $j \neq k$. In purely quantum mechanical terms, the projector P_j represents the question (von Neumann [3]; Jauch [5]) : 'Is the pointer at position M_j?' The observable J has different 'values' at different pointer positions. Since one needs only to distinguish between different pointer positions, any observable J of the above mentioned specifications can serve as a pointer observable.

The phase space function P_j^W corresponding to the projector P_j is supposedly approximated well by a function P_j^{cl} on the phase space Γ of the apparatus A (the $\hbar \to 0$ limit of P_j^W). Now, in Γ, there must be non-overlapping domains D_j corresponding to the pointer positions M_j. In view of the point (iv) in section III, different points in a single domain D_j are not distinguished by the experiment. We can, therefore, take P_j^{cl} to be proportional to the characteristic/indicator function χ_{D_j} of the domain D_j; it follows that the phase space function J^W corresponding to the operator J above is approximated well by the classical pointer observable

$$J^{cl} = \sum_j b'_j \chi_{D_j} \tag{28}$$

where b'_js have properties similar to the b_js above.

The pointer states $\phi_j^{(A)}$ corresponding to the pointer positions M_j are represented by density operators $\rho_j^{(A)}$ supposedly such that

(i) $Tr(\overline{\phi_j^{(A)} P_k}) = \delta_{jk}$;

(ii) the phase space functions $\rho_j^{(A)W}$ corresponding to them are approximated well by the classical phase space density functions $\rho_j^{(A)cl}$ which vanish outside the domain D_j.

We shall take $H_{int} = F \otimes K$ (absorbing the coupling constant in K) where F is the measured quantum observable and K is a suitably chosen apparatus observable . We shall make the usual assumption that, during the measurement interaction, H_{int} is the dominant part of the total Hamiltonian ($H \simeq H_{int}$). The unitary operator U of section II describing the measurement interaction in the von Neumann scheme is now proposed to be replaced by the measurement operator M in supmech which implements the appropriate

canonical transformation on the states of the (S +A) system. It is given by $M \equiv exp[\tau\tilde{\partial}_H]$ where $\tau = t_f - t_i$ is the time interval of measurement interaction and $\tilde{\partial}_H$ is the evolution generator in the supmech Liouville equation [see Eq.(13)].

Assuming, again, that the measurement is ideal and denoting the 'ready state' of the apparatus by $\phi_0^{(A)}$, we have the following analogue of Eq.(1):

$$M(|\psi_j><\psi_j| \otimes \phi_0^{(A)}) = |\psi_j><\psi_j| \otimes \phi_j^{(A)}. \tag{29}$$

Here and in the following developments, we have identified the quantum states of the system S with the corresponding density operators. When the system is initially in the superposition state $|\psi>$ as in section II, the initial and final (S+A)- states are

$$\Phi_{in} = |\psi><\psi| \otimes \phi_0^{(A)}; \quad \Phi_f = M(\Phi_{in}). \tag{30}$$

Note that the 'ready' state may or may not correspond to one of the pointer readings. (In a voltage type measurement, it does; in the Stern-Gerlach experiment with spin half particles, it does not.) For the assignment of the Γ-domain to the 'ready' state, the proper interpretation (which covers both the situations above) of the ready state is 'not being in any of the (other) pointer states'. Accordingly, we assign, to this state, the domain

$$\tilde{D}_0 \equiv \Gamma - \cup_{j\neq 0} D_j \tag{31}$$

where the condition $j \neq 0$ on the right is to be ignored when the 'ready' state is not a pointer state.

We must now take care of the point (iii) of section 3. When the measurement interaction is over, the apparatus, left to itself, will quickly occupy, in any single experiment, a pointer position M_j (depending on the region of the phase space Γ it happens to be in after the measurement interaction). For the ensemble of (S +A) systems described by the initial state Φ_{in}, the final state (after 'settling down') must be of the form

$$\hat{\Phi}_f = \sum_j p_j \rho_j^{(S)} \otimes \phi_j^{(A)} \tag{32}$$

where $\rho_j^{(S)}$ are some states of S. Eq.(32) incorporates the net effect of the processes involved in the 'settling down' of the apparatus. The unknowns

p_j and $\rho_j^{(S)}$ must be determined by identifying the conditions that must be satisfied by the processes involved in the above mentioned 'settling down'.

During the transition from the state Φ_f to $\hat{\Phi}_f$, the change taking place in the system (S+A) is predominantly 'settling down' of the apparatus which, in view of the stability property (iii) above, is not expected to change the expectation value of a pointer observable J. We must have, therefore,

$$\hat{\Phi}_f(A \otimes J) = \Phi_f(A \otimes J) \tag{33}$$

for all system observables A and all pointer observables J of the form (27). It is this condition, based on physical reasoning, which replaces von Neumann's projection postulate in our treatment.

Now, $\Phi_f = \Phi'_f + \Phi''_f$ where

$$\begin{aligned} \Phi'_f &= M\left(\sum_j |c_j|^2[|\psi_j><\psi_j| \otimes \phi_0^{(A)}]\right) \\ &= \sum_j |c_j|^2[|\psi_j><\psi_j| \otimes \phi_j^{(A)}] \end{aligned} \tag{34}$$

(where we have used the fact that the mapping $M \equiv exp[\tau\tilde{\partial}_H]$ on states preserves convex combinations) and

$$\Phi''_f = M\left([\sum_{j\neq k} c_k^* c_j |\psi_j><\psi_k|] \otimes \phi_0^{(A)}\right) \equiv M(R). \tag{35}$$

[Note that R, the operand of M, is not an (S +A)-state; here M has been implicitly extended by linearity to the dual space of the algebra $\mathcal{A}_S \otimes \mathcal{A}_A$.]

We shall now prove that

$$W \equiv \Phi''_f(A \otimes J) \simeq 0. \tag{36}$$

Proof. Transposing the M operation to the observables and adopting the phase space description of the apparatus, we have

$$\begin{aligned} W &= < exp(\tau\tilde{\partial}_H)(R), A \otimes J > = < R, [exp(\tau\partial_H)(A \otimes J)] \\ &= < (\sum_{j\neq k} c_k^* c_j |\psi_j><\psi_k|) \otimes \phi_0^{(A)}, [exp(\tau\partial_H)](A \otimes J) > \\ &= \int_\Gamma d\Gamma \rho_0^{(A)W} \sum_{j\neq k} c_k^* c_j < |\psi_j><\psi_k|, exp(\tau\partial_{H'})(A \otimes J^W) > \end{aligned} \tag{37}$$

where $d\Gamma$ is the phase space volume element, $\rho_0^{(A)W}$ is the Wigner function corresponding to the state $\phi_0^{(A)}$ and $H' = F \otimes K^W$ [see Eq.(22)]. Using equations (9), (17) and (24) above, we have

$$\begin{aligned}\partial_{H'}(A \otimes J^W) &= \{F \otimes K^W, A \otimes J^W\} \\ &= (-i\hbar)^{-1}\left([F,A] \otimes \frac{K^W * J^W + J^W * K^W}{2}\right. \\ &\quad \left. + \frac{FA + AF}{2} \otimes (K^W * J^W - J^W * K^W)\right). \qquad (38)\end{aligned}$$

Given the fact that the apparatus is well described classically, we have $K^W \simeq K^{cl}$ and $J^W \simeq J^{cl}$ to a very good approximation. This gives

$$\partial_{H'}(A \otimes J^W) \simeq (-i\hbar)^{-1} K^{cl} J^{cl} [F, A]$$

which, in turn, implies (recalling the notation $D_F(A) = [F, A]$)

$$\begin{aligned}< |\psi_j >< \psi_k| \quad , \quad & exp(\tau \partial_{H'})(A \otimes J^W) > \\ &= < |\psi_j >< \psi_k|,\ exp(\frac{i\tau}{\hbar} K^{cl} D_F)(A) > J^{cl} \\ &= < \psi_k | exp(\frac{i\tau}{\hbar} K^{cl} D_F)(A) | \psi_j > J^{cl} \\ &= exp[\frac{i\tau}{\hbar} K^{cl}(\lambda_k - \lambda_j)] < \psi_k | A | \psi_j > J^{cl}.\end{aligned}$$

We now have, replacing, in Eq.(37), $\rho_0^{(A)W}$ by its classical approximation $\rho_0^{(A)cl}$,

$$W \simeq \int_{\tilde{D}_0} d\Gamma \rho_0^{(A)cl} \sum_{j \neq k} c_k^* c_j\ exp[\frac{i}{\hbar}(\lambda_k - \lambda_j) K^{cl} \tau] J^{cl} < \psi_k | A | \psi_j > . \qquad (39)$$

Let

$$< K^{cl} >_0 \equiv \int_{\tilde{D}_0} K^{cl} \rho_0^{(A)cl} d\Gamma \qquad (40)$$

(the mean value of K^{cl} in the domain $\tilde{D}_0$; we shall give an argument below showing that it is nonzero). Putting $K^{cl} =< K^{cl} >_0 s$, taking s to be one of the integration variables and writing $d\Gamma = ds d\Gamma'$, we have

$$W \simeq \int_{\tilde{D}_0} ds d\Gamma' \rho_0^{(A)cl} \sum_{j \neq k} c_k^* c_j\ exp[\frac{i}{\hbar} \eta_{jk} s] J^{cl} < \psi_k | A | \psi_j > \qquad (41)$$

where

$$\eta_{jk} = (\lambda_k - \lambda_j) < K^{cl} >_0 \tau. \tag{42}$$

Note that s is a real dimensionless variable with a bounded domain of integration [see remark (iii) below].

We shall now argue that, for $j \neq k$,

$$|\eta_{jk}| >> \hbar. \tag{43}$$

[This is not obvious; when F is a component of spin, for example, the quantity $(\lambda_k - \lambda_j)$ is a scalar multiple of $\hbar$.] To this end, we invoke the apparatus feature (v) of section III. A reasonable procedure for formulating a criterion for macroscopic distinguishability of different pointer positions would be to identify a quantity of the dimension of action which could be taken as characterizing the physical separation between two different pointer positions and show that its magnitude is much larger than $\hbar$. The objects η_{jk} (for $j \neq k$) are quantities of this type. A simple way of seeing this is to treat Eq.(43) as the time-energy uncertainty inequality $|\Delta E \Delta t| >> \hbar$ where $\Delta t = \tau$ and ΔE is the difference between the energy values corresponding to the apparatus locations in two different domains D_j and D_k in Γ. Recalling that $H \simeq H_{int}$ during the relevant time interval, we have

$$\Delta E \simeq (\lambda_k - \lambda_j) < K^{cl} >_0 . \tag{44}$$

[See the remark (ii) below.] The inequality (43) then follows from the assumed macroscopic distinguishability of different pointer positions. This assumption along with the argument above also implies $< K^{cl} >_0 \neq 0$ as promised above.

The large fluctuations implied by Eq.(43) wipe out the integral in Eq.(41) giving $W \simeq 0$ as desired. □

Remarks. (i) For an argument, starting from the condition of macroscopic distinguishability of pointer positions and arriving at the time-energy uncertainty inequality in the context of the Stern-Gerlach experiment, see (Gottfried [15]).

(ii) How does one justify the appearance of the mean value of K^{cl} in the 'ready' state in the expression for ΔE in the energy time inequality above ? A plausible answer is this : Since the apparatus is initially in the 'ready'

state and since K appears in H_{int},, it is the quantity $< K^{cl} >_0$ which will, at the classical level, be effective in determining the probabilities of transitions to the various domains D_j.
[A more refined argument : Suppose, at time $t = t_i$, the system point of the apparatus A, considered as a classical system, in the phase space Γ is $\xi_0 \in \tilde{D}_0$. With the system in the initial state $|\psi_k >$, the effective classical Hamiltonian is $H^{cl(k)} = \lambda_k K^{cl}$. After the measurement interaction, at $t = t_f$, we have the system point of A at $\xi(t_f) = \xi_{0k} \in D_k$. For the quantity ΔE considered above, a good estimate is

$$\Delta E \simeq \int d\Gamma(\xi_0)\rho_0^{(A)cl}(\xi_0)[\lambda_k K^{cl}(\xi_{0k}) - \lambda_j K^{cl}(\xi_{0j})]. \tag{45}$$

But the Hamiltonians $H^{cl(k)}$ and $H^{cl(j)}$ conserve the quantity K^{cl}. This gives $K^{cl}(\xi_{0k}) = K^{cl}(\xi_0) = K^{cl}(\xi_{0j})$, hence Eq.(44).]

(iii) Physical quantities related to the apparatus must, in their classical description, be bounded functions on Γ. (Even observables like the Cartesian components of position or momentum of macroscopic parts/components of the apparatus must vary in finite intervals.) Boundedness of the domain of integration of the variable s now follows from the relation $|s|_{max} = |(< K^{cl} >_0)^{-1} K^{cl}|_{max}$. [*Note.* In the example in the next section, the dimensionless variable u, playing the same role as s here, has domain of variation of length of order 1. In the general case, let $s_1 \leq s \leq s_2$. If $s_2 - s_1 \leq 2\pi$, no further argument is necessary. If $s_2 - s_1 > 2\pi$, put $s = (s_2 - s_1)w$; now the integration variable w is similar to u and the additional factor $(s_2 - s_1)$ in the exponent is welcome.]

Equations (33), (32) and (36) now give

$$\begin{aligned} 0 &\simeq (\hat{\Phi}_f - \Phi'_f)(A \otimes J) \\ &= \sum_j \phi_j^{(A)}(J) Tr(\overline{[p_j \rho_j^{(S)} - |c_j|^2 |\psi_j >< \psi_j|]A}) \\ &= \sum_j b_j Tr(\overline{[p_j \rho_j^{(S)} - |c_j|^2 |\psi_j >< \psi_j|]A}) \end{aligned}$$

which must be true for arbitrary b_j in Eq.(27) satisfying the stated condition. This gives

$$Tr(\overline{[p_j \rho_j^{(S)} - |c_j|^2 |\psi_j >< \psi_j|]A}) = 0$$

for all j and all system observables A and, therefore, for all j,

$$p_j \rho_j^{(S)} = |c_j|^2 |\psi_j >< \psi_j|.$$

Finally, therefore, we have $\hat{\Phi}_f = \Phi'_f$ which is precisely the state obtained from Φ_f by applying the von Neumann projection.

This completes the derivation of the von Neumann projection rule. This has been obtained through straightforward physics; there is no need to give any ad hoc prescriptions. The derivation makes it clear as to the sense in which this reduction rule should be understood : it is a prescription to correctly take into consideration the effect of the 'settling down' of the apparatus after the measurement interaction for obtaining the final state of the system (S + A) observationally constrained as in items (iv) and (v) of section 3.

Eq.(41), followed by the reasoning above, represents, in a *live* form, the operation of environment-induced decoherence. To see this, note that, the domain $\tilde{D}_0$ may be taken to represent the internal environment of the apparatus. With this understanding, the mechanism wiping out the unwanted quantum interference terms is, indeed, the environment-induced decoherence. In the treatment presented here this mechanism becomes automatically operative. (Even the external environment can be trivially included by merely saying that the system A above represents 'the apparatus and the external environment'.)

6. Example : The Stern-Gerlach Experiment

As an illustration of the automatic appearance of the decoherence mechanism in the supmech based treatment of quantum measurements presented in the previous section, we consider the Stern-Gerlach experiment (Busch, Grabowski and Lahti [16]; Omnes [6]; Gottfried [15]; Cohen- Tannoudji, Diu and Laloë [17]) with, say, silver atoms (which means spin s = $\frac{1}{2}$). A collimated beam of (unpolarized) silver atoms is made to pass through inhomogeneous magnetic field after which the beam splits into two beams corresponding to atoms with $S_z = \pm\frac{\hbar}{2}$. The spin and magnetic moment operators of an atom are $\mathbf{S} = \frac{\hbar}{2}\sigma$ and $\mu = g\mathbf{S}$ (where g is the magnetogyric ratio). Let the magnetic field be $\mathbf{B}(\mathbf{r}) = B(z)\mathbf{e_3}$ (in obvious notation). [Refinements (Potel et al [18]) introduced to ensure the condition $\bigtriangledown.\mathbf{B} = 0$ do not affect the essential results obtained below.] We have

$$H_{int} = -\mu.\mathbf{B} = -gB(z)S_3. \tag{46}$$

The force on an atom, according to Ehrenfest's theorem, is

$$\mathbf{F} = -\bigtriangledown < -\mu.\mathbf{B} >= g\frac{dB(z)}{dz} < S_3 > \mathbf{e_3} \tag{47}$$

where the average is taken in the quantum state of the atom. During the experiment, the internal state of the atom remains unchanged (to a very good approximation); only its center of mass **r** and spin **S** have significant dynamics. In this experiment, S_3 is the measured quantum observable and **r** acts as the operative apparatus variable.

Let us assume that the beam initially moves in the positive x-direction, the pole pieces are located in the region $x_1 \leq x \leq x_2$ and the detectors located in the plane $x = x_3 > x_2$ (one each in the regions $z > 0$ and $z < 0$; these regions contain the emergent beams of silver atoms corresponding, respectively, to $S_3 = +\frac{\hbar}{2}$ and $S_3 = -\frac{\hbar}{2}$). We have, in the notation used above, $F = S_3$ and K = - g B(z). Assuming the experiment to start when the beam reaches at $x = x_1$, the phase space of the apparatus is

$$\Gamma = \{(x, y, z, p_x, p_y, p_z) \in \mathbb{R}^6; x \geq x_1, *\} \tag{48}$$

where * indicates the restriction that, for $x_1 \leq x \leq x_2$, the space available for the movement of atoms is the one between the two pole pieces. For the order of magnitude calculation below, we shall ignore the shape of the pole pieces and take * to imply $z_1 \leq z \leq z_2$.

The domains D_1 and D_2 corresponding to the two pointer positions are

$$\begin{aligned} D_1 &= \{(x, y, z, p_x, p_y, p_z) \in \Gamma; x > x_2, p_z > 0\} \\ D_2 &= \{(x, y, z, p_x, p_y, p_z) \in \Gamma; x > x_2, p_z < 0\}; \end{aligned}$$

the domain $\tilde{D}_0 = \Gamma - (D_1 \cup D_2)$. For simplicity, let us take $B(z) = b_0 + b_1 z$ where b_0 and b_1 are constants. For $j \neq k$, we have $\lambda_j - \lambda_k = \pm\hbar$. The relevant integral is [see Eq.(41) above]

$$I = \int_{z_1}^{z_2} dz(...)exp[\pm\frac{i}{\hbar}\mu b_1 z\tau] \tag{49}$$

where $\mu = g\hbar$. Putting $z = (z_2 - z_1)u$, the new integration variable u is a dimensionless variable taking values in a domain of length of order one. The quantity of interest is

$$|\eta| = \mu|b_1|(z_2 - z_1)\tau. \tag{50}$$

According to the data in (Cohen-Tannoudji, Diu and Laloë [17]) and (Goswami [19]; problem 4.6), we have (v_x is the x- component of velocity of the silver atom)

$$|b_1| \sim |\frac{dB}{dz}| \sim 10^5 \; gauss/cm$$
$$z_2 - z_1 \simeq 1 \; mm, \quad v_x \sim 500 \; m/sec$$
$$x_2 - x_1 = 3 \; cm, \quad x_3 - x_2 = 20 \; cm$$

This gives

$$\tau \sim \frac{x_3 - x_1}{v_x} \sim 5 \times 10^{-4} sec.$$

Denoting the Bohr magneton by μ_b and putting $\mu \sim \mu_b \simeq 0.9 \times 10^{-20}$ erg/gauss, we have $|\eta| \sim 10^{-19}$erg-sec. With $\hbar \simeq 1.1 \times 10^{-27}$ erg-sec, we have, finally, $(|\eta|/\hbar) \sim 10^8$, confirming the strong suppression of the undesirable quantum interferences.

7. Comparison with the Traditional Decoherence Program

In the traditional decoherence program, one invokes the interaction of the system (S + A) with the environment (to be denoted as $\mathcal{E}$), treat the combined system (S + A + $\mathcal{E}$) as in section 2 up to the pre-measurement level of Eq.(2) and then take the partial trace of the density operator of (S + A + $\mathcal{E}$) over the environment $\mathcal{E}$ to obtain the reduced density operator for the system (S + A). Making some plausible assumptions about the states of the environment (which can be verified in concrete model calculations), one can show that the reduced density operator for (S+A) has the desired form (4) incorporating the von Neumann projection. Taking trace over $\mathcal{E}$ is interpreted as ignoring uncontrolled and unmeasured degrees of freedom; this is, supposedly, similar to the procedure of deriving the probability $\frac{1}{2}$ for 'heads' as well for 'tails' in the experiment of tossing a fair coin by averaging over the uncontrolled and unmeasured degrees of freedom of the environment of the coin.

A critical look at these developments (see Bub [10], Adler [11]), however, shows that there are loopholes. As argued by Bub, the two averaging procedures are not on the same footing. In the coin toss experiment, when, ignoring the environment, we claim that, the probability of getting 'heads'

in a particular toss of the coin is $\frac{1}{2}$, we can also claim that we do, in fact, get *either* 'heads' or 'tails' on each particular toss. A definite outcome can be predicted if we take into consideration the environmental effects and details of initial conditions of the toss. In the treatment of a quantum measurement in the traditional decoherence program as outlined above, however, we cannot claim that, taking the environment into consideration, a definite outcome of the experiment will be predicted. In fact, taking the environment will give us back a troublesome equation of the form (2) [with A replaced by (A + $\mathcal{E}$)] which is obtined in a von Neumann type treatment of the system (S + A + $\mathcal{E}$).

The problem really lies with some unsatisfactory features and inadequacies in the treatment of the apparatus in von Neumann's treatment of measurements. (See section 3.) In the present work, the physics of the apparatus has been taken seriously and the points mentioned in section 3 have been properly incorporated in the supmech based treatment of measurements in section 5. The decoherence effects arise because observations on the apparatus are restricted to macroscopically distinguishable pointer readings which inevitably leads to averaging over the inoperative degrees of freedom (internal environment) [see Eq.(41).] Here is an 'averaging over uncontrolled and unmeasured degrees of freedom' which is arising in a consistent scheme of mechanics; there are obviously no problems of consistency or logical coherence here.

Another important positive feature of the present treatment is that the paradoxical aspect of the 'Heisenberg cut' or Bohr's 'quantum-classical divide' has been gotten rid of in the most natural way — by treating the apparatus carefully as a quantum system approximated well by a classical one.

8. The Eighth Axiom of the Supmech Program

A provisional set of seven axioms, underlying the plan to do 'all physics' in the framework of a noncommutative symplectic geometry based universal mechanics adopted in this series of papers (*the supmech program*), was presented in section 5 of II. The use of the word 'provisional' reflects the expectation that a stage may come when, after achieving some successes, a more compact set of axioms (which may themselves be 'provisional' at a higher level) is found more suitable. Till such a stage is reached, the list of provisional axioms is expected to increase with additions in coverage of the

program. The new assumptions made in the present work are being listed below as the eighth provisional axiom.

A8. *Measurements.* In a measurement involving a 'measured system' S and apparatus A, the following items hold :

(a) Both S and A are standard quantum systems (as defined in section 3.2 of II; it means that that the system algebra is a noncommutative $\star$-algebra generated by a finite number of fundamental observables and the unit element).

(b) The supmech Hamiltonian system $(\mathcal{A}^{(A)}, \mathcal{S}_1^{(A)}, \omega^{(A)}, H^{(A)})$ corresponding to the apparatus admits an equivalent (in the sense of section IV C) phase space realization (in the Weyl-Wigner-Moyal scheme) $(\mathcal{A}_W^{(A)}, \mathcal{S}_{1W}^{(A)}, \omega_W^{(A)}, H_W^{(A)})$.

(c) Elements of $\mathcal{A}_W^{(A)}$ and $\mathcal{S}_{1W}^{(A)}$ appearing in the description of dynamics of the coupled system (S+A) admit $\hbar \to 0$ limits and are approximated well by these limits.

(d) The pointer positions of the apparatus have the stability property as stated in section III [item (iii)]. Different pointer positions have mutually disjoint stability domains in the phase space of the apparatus. When the apparatus, after some interaction, is left to itself with its 'system point' (the point in the apparatus phase space representing its instantaneous classical state) in one of the stability domains, it will eventually settle down to the corresponding pointer position.

(e) Different pointer positions are macroscopically distinguishable [the macroscopic distinguishability can be interpreted, for example, in terms of an energy-time uncertainty product inequality ($\Delta E \Delta t >> \hbar$) relevant to the experimental situation].

(f) observations on the apparatus are restricted to readings of the output devices (pointers).

9. Concluding Remarks

1. The central message of the present work is this : 'In the theoretical treatment of measurement on a quantum system, the apparatus must be properly treated as a quantum *system* approximated well by a classical one.' The main body of the paper is devoted to just doing this job sensibly.

We have seen in operation the general plan (II, sections 4 and 6) for dealing with situations involving quantum systems approximated well by classical ones : Start with the quantum system (treating it as a supmech Hamiltonian system), transform it to an equivalent supmech Hamiltonian system, employing the Weyl-Wigner-Moyal formalism and then introduce the classical approximations of the relevant phase space functions. This made it possible (and smooth going) the treatment of the apparatus along the above mentioned lines.

This reduces the problem of 'quantum-classical divide' in the foundations of quantum mechanics to a non-problem — there is no need for such an ad hoc divide now.

2. It is worth (re-)emphasizing that, in the theoretical treatment of quantum measurements, if the physics of the apparatus is treated adequately (with due attentions to points mentioned in section III), the decoherence effects needed to wipe off the unwanted quantum interferences appear automatically. The sight of Eq.(41) where one can see the operation of the decohering effect of averaging over the passive region of the apparatus phase space (which can be interpreted as the effect of the 'internal environment' of the apparatus) in live action, should please theoreticians. The incorporation of the external environment (for the restricted purpose of realizing von Neumann reduction) has been reduced to a matter of less than two lines : just saying that the symbol 'A' now stands for 'the apparatus and the external environment'.

It is easily checked that the arguments against the traditional decoherence program [7] do not apply in the present case. Here the desired decoherence effects appear as a matter of course in a consistent scheme of mechanics — no questionable ad-hoc procedures were employed to obtain those effects.

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